NAVIER STOKES EQUATIONS

2D Global Regularity and Difficulties in 3D

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Contents

1. Introduction

This article will focus on the Navier-Stokes equations, which describe incompressible flows of homogeneous fluids in all of space $\mathbb{R}^N$. First, we will introduce the equations, state some preliminaries and summarize and sketch the proof of local-in-time existence of solutions (in both two and three dimensions). Then, we will use this to prove global regularity in two dimensions (via the Beale-Kato-Majda criterion using the vorticity). Our reference for this will be the first three chapters of "Vorticity and incompressible flow" by Bertozzi and Majda ([1]).

At the end, we will also discuss the difficulties in global regularity in three dimensions, guided by Terence Tao's article [2]. We will also discuss some related ideas from Tao in [4] on what strategies may or may not work in attempting to prove global regularity in three dimensions or attempting to find solutions that blow up in finite time.

The Navier-Stokes equations are as follows:

$$\frac{Dv}{Dt} = -\nabla p + \nu \Delta v \tag{1}$$

$$\text{div } v = 0, \qquad (x,t) \in \mathbb{R}^N \times [0,\infty) \tag{2}$$

$$v|_{t=0} = v_0, \qquad x \in \mathbb{R}^N \tag{3}$$

Here $v(x,t) \equiv (v^1, v^2, \dots, v^N)^T$ is the fluid velocity, $p(x,t)$ is the scalar pressure, D/Dt is the convective derivative (i.e., the derivative along particle trajectories) defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{N} v^j \frac{\partial}{\partial x_j}, \qquad (2.4)$$

and $\nu \geq 0$ is the kinematic viscosity (if $\nu = 0$, this reduces to the Euler equations). These equations follow from physical laws of conservation of momentum and mass and describe the velocity field of an incompressible viscous fluid. The so far unsolved Millenium problem for the Navier-Stokes equations asks if solutions to these equations in three dimensions remain smooth for all time, or if they can develop a singularity in finite time from smooth initial data.

The approach taken in [1] to study these equations relies on the *vorticity*. Given a vector field v in three dimensions, its vorticity ω is defined as

$$\omega = \text{curl } \mathbf{v} \equiv \left(\frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3}, \frac{\partial v^1}{\partial x_3} - \frac{\partial v^3}{\partial x_1}, \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} \right)^t.$$

If we take the curl of the Navier-Stokes equations, we can get an evolution equation for ω. Note that there is no $\omega \cdot \nabla v$ term in 2D flows, since there the velocity field is $v = (v^1, v^2, 0)^\top$, independent of x_3, and the vorticity $\omega = (0, 0, v^2_{x_1} - v^1_{x_2})^\top$ has only a third component, so $\omega \cdot \nabla v = \omega^3 \partial_3 v \equiv 0$. So in two dimensions, we get the following scalar vorticity equation:

$$\frac{D\omega}{Dt} = \omega_t + v \cdot \nabla \omega = \nu \Delta \omega. \qquad (4)$$

2. Preliminaries, local-in-time existence

In this section, I will summarize the key ideas used in [1] to prove local-in-time existence of solutions to the Navier-Stokes equations (in both two and three dimensions), and some ideas that will be essential for the proof of global regularity in two dimensions.

[1] says that it is a reasonable assumption that velocity fields in three dimensions decay fast at infinity, enough so $v \in L^2(\mathbb{R}^3)$. However, it turns out that this is not a reasonable assumption in two dimensions; even velocity fields with vorticity that is C_c^∞ doesn't guarantee being globally in $L^2(\mathbb{R}^2)$. In fact, a 2D incompressible Biot-Savart velocity field with smooth ω of compact support has finite kinetic energy if and only if $\int_{\mathbb{R}^2} \omega(x,t)\ \mathrm{d}x = 0$. Instead of requiring globally finite kinetic energy, it turns out that we can decompose v into the sum of two parts, with one having globally finite kinetic energy and another being a radially symmetric exact (eddy) solution.

Proposition 2.0.1 (Radial Energy Decomposition)**.** *Any 2D Biot-Savart velocity $v = K_2 * \omega$, where $K_2(x) = (-x_2, x_1)/(2\pi|x|^2)$ and $\omega \in C_c^\infty(\mathbb{R}^2)$, has a radial-energy decomposition of the form*

$$v(x) = u(x) + \bar{v}(x)$$

$$\int |u(x)|^2 dx < \infty, \quad \textit{div } u = 0$$

$$\bar{v}(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} |x|^{-2} \int_0^{|x|} s\bar{\omega}(s)ds$$

where $\bar{\omega}(r)$ is a smooth compactly supported radially symmetric vorticity with $\int_{\mathbb{R}^2} \bar{\omega}\, dx = \int_{\mathbb{R}^2} \omega\, dx$. For time-dependent flows, choose $\bar{\omega}_0$ this way and set $\bar{\omega}(t) = e^{\nu t \Delta}\bar{\omega}_0$. When $\nu > 0$, the corresponding $\bar{v}(x,t)$ is an exact viscous eddy solution of the Navier-Stokes equation; when $\nu = 0$, it is stationary and solves Euler.

The general strategy [1] uses to prove local-in-time existence is as follows: first, we find an approximate (regularized) equation and approximate solutions such that proving existence for all time is not very difficult, and the solutions satisfy an energy estimate. Then, we pass to a limit to get a solution to Navier-Stokes. We will summarize this in the next two subsections and sketch some of the main ideas. We will summarize the proofs in the three dimensional case; the proofs for the two dimensional case are essentially the same, except we would have to use the radial finite-energy decomposition.

2.1. Sketch of global existence for regularized Navier-Stokes

In order to create the regularized equation, we need two ingredients: mollifiers, and Leray's projection operator. Given a radial function

$$\rho(|x|) \in C_0^\infty(\mathbb{R}^N), \quad \rho \geq 0, \quad \int_{\mathbb{R}^N} \rho\, dx = 1, \tag{5}$$

we define the mollification $\mathcal{J}_\epsilon v$ of function $v \in L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, by

$$(\mathcal{J}_\epsilon v)(x) = \epsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{x-y}{\epsilon}\right) v(y)\, dy, \quad \epsilon > 0, \tag{6}$$

then we know by convolution properties that $\mathcal{J}_\epsilon v$ is C^∞, and also that for $v \in C^0(\mathbb{R}^N)$, $\mathcal{J}_\epsilon v \to v$ uniformly on compact sets.

Then, let V^s be the Banach space of divergence-free functions in $H^s(\mathbb{R}^N)$, i.e.

$$V^s = \{v \in H^s(\mathbb{R}^N) : \text{div } v = 0\}.$$

[1] then defines Leray's projection operator P onto V^s, which commutes with derivatives and mollifiers. Using all this, [1] formulates the regularized Navier-Stokes equations as the following ODE in V^s:

$$\frac{dv^\epsilon}{dt} = F_\epsilon(v^\epsilon), \quad v^\epsilon|_{t=0} = v_0, \tag{7}$$

where

$$F_\epsilon(v^\epsilon) = \nu \mathcal{J}_\epsilon^2 \Delta v^\epsilon - P\mathcal{J}_\epsilon[(\mathcal{J}_\epsilon v^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon v^\epsilon)] = F_\epsilon^1(v^\epsilon) - F_\epsilon^2(v^\epsilon). \tag{8}$$

First, [1] proves the existence of regularized solutions to (7) using Picard's Theorem on Banach Spaces. As needed for this theorem, F_ϵ maps $V^m \to V^m$ since div $v^\epsilon = 0$, P projects onto divergence-free fields, and $\mathcal{J}_\epsilon$ smooths and commutes with derivatives. Also, F_ϵ is locally Lipschitz continuous on any open set such as $O^M = \{v \in V^m : \|v\|_m < M\}$, as proven by [1] in Proposition 3.6. Then Picard's Theorem gives a unique solution $v^\epsilon \in C^1([0, T_\epsilon); V^m)$

given any initial $v_0 \in V^m$, choosing $M > \|v_0\|_m$.

[1] is then also able to prove an energy estimate (i.e. bound on the $H^0 = L^2$ norm): on any $[0,T]$ where this solution v^ϵ belongs to $C^1([0,T];V^0)$, we have

$$\sup_{0\le t\le T} \|v^\epsilon(t,\cdot)\|_0 \le \|v_0\|_0 . \tag{9}$$

The proof just involves taking the L^2 inner product of (7) with v^ϵ, integrating by parts, and using the properties of mollifiers and the projection operator P. In fact, the specific choice of regularization in (8) (e.g. the "extra" J_ϵ's) are chosen so this integration by parts works as needed.

Then, [1]'s Theorem 3.3 says that solutions to the ODE can be continued in time provided we prove an a-priori bound on $\|v^\epsilon(\cdot,t)\|_m$, which is easy to do using the energy bound, the mollifier estimate $\|\nabla \mathcal{J}_\epsilon v^\epsilon\|_{L^\infty} \le C_\epsilon \|v_0\|_0$, the H^m energy estimate below, and Gronwall's Lemma.

2.2. Sketch of local-in-time existence

Now, given global existence and uniqueness of solutions to the regularized equation, we want to show that there's a time interval $[0,T]$ and subsequence v^ϵ that converges to a limit function $v \in C([0,T];C^2) \cap C^1([0,T];C)$ solving the Navier-Stokes equation. In fact, given an initial condition $v_0 \in V^m$ for integer $m > \frac{N}{2} + 2$, we will show exactly this for any $0 < T < \frac{1}{c_m\|v_0\|_m}$ (with $v_0 = 0$ giving the zero solution).

First, we need the following energy estimate.

Proposition 2.2.1 (H^m Energy Estimate, Proposition 3.7 from [1])**.** *For $v_0 \in V^m$, the unique regularized solution $v^\epsilon \in C^1([0,\infty);V^m)$ to 7 satisfies*

$$\frac{d}{dt}\frac{1}{2}\|v^\epsilon\|_m^2 + \nu\|\nabla \mathcal{J}_\epsilon v^\epsilon\|_m^2 \le c_m|\nabla \mathcal{J}_\epsilon v^\epsilon|_{L^\infty}\|v^\epsilon\|_m^2.$$

The proof here involves taking the derivative D^α (for $|\alpha| \le m$) of the regularized equation, and then taking the L^2 inner product with $D^\alpha v^\epsilon$, then integrating by parts or using the divergence theorem. Then, we sum over $|\alpha| \le m$ and simplify.

Using this and the Sobolev inequality embedding $H^{s+k}(\mathbb{R}^N), s > N/2$ into $C^k(\mathbb{R}^N)$, we can show that for $m > N/2 + 1$,

$$\frac{d}{dt}\|v^\epsilon\|_m \le c_m|\mathcal{J}_\epsilon \nabla v^\epsilon|_{L^\infty}\|v^\epsilon\|_m \le c_m\|v^\epsilon\|_m^2,$$

so for all ϵ we have

$$\sup_{0\le t\le T}\|v^\epsilon\|_m \le \frac{\|v_0\|_m}{1 - c_mT\|v_0\|_m} = \|v_0\|_m + \frac{\|v_0\|_m^2 c_m T}{1 - c_mT\|v_0\|_m}. \tag{10}$$

Therefore, as long as $T < \frac{1}{c_m\|v_0\|_m}$, $\{v^\epsilon\}$ is uniformly bounded in $C([0,T];H^m)$. Then, the time derivatives $\left\{\frac{dv^\epsilon}{dt}\right\}$ are uniformly bounded in H^{m-2} using this, since $\frac{dv^\epsilon}{dt} = F_\epsilon(v^\epsilon)$.

We then show that the solutions v^ϵ form a Cauchy sequence in the low norm $C([0,T];L^2(\mathbb{R}^3))$. Specifically, we show via integration by parts, mollifier properties, and Sobolev inequality that

$$\sup_{0<t<T} \left\| v^\epsilon - v^{\epsilon'} \right\|_0 \leq C \max(\epsilon, \epsilon'),$$

which then gives us the existence of a v such that $\sup_{0\leq t\leq T} \|v^\epsilon - v\|_0 \leq C\epsilon$. Since the v^ϵ are uniformly bounded in a high norm (i.e. Equation 10), we can apply a Sobolev interpolation lemma (between $H^s(\mathbb{R}^N)$ and $H^{s'}(\mathbb{R}^N)$ for $0 < s' < s$) on the difference $v^\epsilon - v$ to show that we have strong convergence in the lower norms. Then since

$$v_t^\epsilon = \nu \mathcal{J}_\epsilon^2 \Delta v^\epsilon - P\mathcal{J}_\epsilon[(\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla(\mathcal{J}_\epsilon v^\epsilon)],$$

we have that v_t^ϵ converges in $C([0,T], C(\mathbb{R}^3))$ to $\nu\Delta v - P(v \cdot \nabla v)$. Since $v^\epsilon \to v$, the distributional limit of v_t^ϵ is v_t, so v is a classical solution of the Navier-Stokes equations, with $\nabla p = -(I - P)(v \cdot \nabla v)$. Uniqueness follows from the L^2 energy estimate for the difference of two solutions. Then, [1] proves that the solution is in $C([0,T]; H^m) \cap C^1([0,T]; H^{m-2})$; the proof uses that the forward diffusion in Navier-Stokes ($\nu > 0$) introduces smoothing into the evolution, and uses time reversibility instead for Euler ($\nu = 0$).

In the end, since only initial data controlled the time of existence, i.e. $T < \frac{1}{c_m \|v_0\|_m}$, as long as $\|v(\cdot, t)\|_m$ remains bounded, we can continue the solution in time (i.e. at time T, just choose $v(x, T)$ as the initial data and repeat).

3. Global existence/regularity in two dimensions

The theorem we prove will be a two dimensional variant of Theorem 3.6 in [1], which states in three dimensions that: given an initial velocity $v_0 \in V^m$, integer $m > N/2 + 2$, so that there exists a classical solution $v \in C([0,T_*); V^m) \cap C^1([0,T_*); V^{m-2})$ to the Euler or Navier-Stokes equation on its maximal interval, if for any finite $T > 0$ there exists $M_T > 0$ such that the vorticity $\omega = \text{curl } v$ satisfies $\sup_{0<t<\min(T,T_*)} \int_0^t \|\omega(\cdot,\tau)\|_{L^\infty} d\tau \leq M_T$ then the solution v exists globally in time, $v \in C([0,\infty); V^m) \cap C^1([0,\infty); V^{m-2})$. I believe this theorem is referred to as the "Beale-Kato-Majda criterion," as seen in [3]. In [1], this theorem is proven in three dimensions, and Corollary 3.3 states without proof the result in two dimensions, but also states that the proof can be derived from the proof of Theorem 3.6 by using the radial-energy decomposition described earlier.

Before doing this, will need the following propositions, which we will use without proof:

Proposition 3.0.1 (2D H^m Energy Estimate, variant of Proposition 3.7A from [1])**.** *Given an initial 2D velocity $v_0(x)$ and corresponding local-finite energy decomposition $u_0(x) + \overline{v_0}(x)$, $u_0 \in V^m(\mathbb{R}^2)$, $m \in \mathbb{Z}^+ \cup \{0\}$, with smooth compactly supported radial $\bar{\omega}_0 = \text{curl}\, \bar{v}_0$, the regularized solution $u^\epsilon \in C^1([0,T_\epsilon); V^m)$ to Eq. 3.74 in [1] satisfies*

$$\frac{d}{dt}\|u^\epsilon\|_m \leq c_m \left(\|\nabla \mathcal{J}_\epsilon u^\epsilon\|_{L^\infty} + B_m(t) \right) \|u^\epsilon\|_m, \tag{11}$$

where $B_m(t) = \sum_{1\leq|\alpha|\leq m+1} \|D^\alpha \bar{v}(t)\|_{L^\infty}$ keeps the dependence on higher derivatives of the background explicit.

Proposition 3.0.2 (2D Potential Theory Estimate, analogue of Proposition 3.8 from [1])**.** *Let* $u : \mathbb{R}^2 \to \mathbb{R}^2$ *be divergence-free, with finite energy and* $\zeta = curl\ u \in L^2 \cap L^\infty, u \in H^m(\mathbb{R}^2)$ *for* $m > 2$*. Then there exists a constant* $C(m)$ *s.t.*

$$\|\nabla u\|_{L^\infty} \leq C(1 + \ln^+ \|u\|_{H^m})(1 + \|\zeta\|_{L^\infty}), \tag{12}$$

where $\ln^+ r = \max(0, \ln r)$ *for* $r > 0$*, and* $\ln^+ 0 = 0$*.*

Now, here is the main theorem:

Theorem 3.0.3 (Global existence in two dimensions)**.** *Let* v_0 *be an initial 2D velocity field with locally finite-energy decomposition* $v_0 = u_0 + \bar{v}_0$*, with* $u_0 \in V^m(\mathbb{R}^2)$*, integer* $m > 3$*, and smooth compactly supported radial* $\bar{\omega}_0(r) = curl\ \bar{v}_0$*. Then, there exists for all time a unique classical solution*

$$v(x,t) = u(x,t) + \bar{v}(x,t)$$

to the 2D Euler (respectively, Navier-Stokes) equation, with $u \in C([0,T]; V^m) \cap C^1([0,T]; V^{m-2})$ *on any finite time interval* $[0,T]$ *and* $\bar{v}(x,t)$ *an exact inviscid (respectively, viscous) eddy solution. Smooth initial data give a smooth solution; for* $\nu > 0$*, the solution is smooth for* $t > 0$*.*

Proof. (Note: *This proof is largely based on the proof of the three dimensional Theorem 3.6 in [1] on Page 115, except I've attempted to adapt it for the two dimensional case.*)

First, note from the previous section that, by our construction for a local-in-time solution to Euler or Navier-Stokes with a radial-energy decomposition, we can continue the solution as long as $\|u\|_m$ stays bounded: if $T_* < \infty$ is the maximal time of existence, then $\limsup_{t \uparrow T_*} \|u(\cdot,t)\|_m = \infty$, provided the background derivatives remain bounded. So it suffices to prove an a-priori bound for $\|u(\cdot,t)\|_m$ under these assumptions.

We start by applying the same energy calculation as in equation (11) to the perturbation equation $u_t + u \cdot \nabla u + \bar{v} \cdot \nabla u + u \cdot \nabla \bar{v} = -\nabla(p - \bar{p}) + \nu \Delta u$. For Euler or Navier-Stokes, this gives us the following 2D energy estimate (in integrated form for H^m solutions, by approximation):

$$\frac{d}{dt} \|u\|_{H^m} \leq C_m \left(\|\nabla u\|_{L^\infty} + B_m(t)\right) \|u\|_{H^m}. \tag{13}$$

To use Gronwall's lemma to control $\|u\|_{H^m}$, we need to bound $\|\nabla u(t)\|_{L^\infty} + B_m(t)$. Equation (12) gives us a bound for the first term. For the background, note that

$$\bar{v}(x,t) \ = \ \frac{(-x_2, x_1)}{|x|^2} \int_0^{|x|} s\, \bar{\omega}(s,t)\, ds,$$

so if we differentiate we get

$$|\nabla \bar{v}(x,t)| \leq C \frac{1}{|x|^2} \int_0^{|x|} s|\bar{\omega}(s,t)|\, ds + C|\bar{\omega}(|x|,t)|$$

via the product rule, so $\|\nabla\bar{v}(t)\|_{L^\infty} \le C\|\bar{\omega}(t)\|_{L^\infty}$. Since $\bar{\omega}(t) = e^{\nu t\Delta}\bar{\omega}_0$ and $\bar{v}(t) = e^{\nu t\Delta}\bar{v}_0$, the heat semigroup bounds (or stationarity for $\nu = 0$) give

$$\|\bar{\omega}(t)\|_{L^\infty} \le \|\bar{\omega}_0\|_{L^\infty}, \qquad B_m(t) \le B_m(0) < \infty.$$

Writing $\omega = \operatorname{curl} v$ and $\zeta = \operatorname{curl} u = \omega - \bar{\omega}$, we get in total that

$$\|\nabla u\|_{L^\infty} + B_m(t) \le C(1 + \ln^+\|u\|_m)(C_0 + \|\omega\|_{L^\infty}), \qquad C_0 = 1 + \|\bar{\omega}_0\|_{L^\infty} + B_m(0).$$

Then, let $G(t) = \|u(t)\|_m + e$ (so $\ln G > 0$). Then we find that

$$\begin{aligned}\frac{d}{dt}G(t) = \frac{d}{dt}\|u\|_{H^m} &\le C(1 + \ln^+\|u\|_{H^m})(C_0 + \|\omega\|_{L^\infty})\|u\|_{H^m}\\ &\le C(1 + \ln G)(C_0 + \|\omega\|_{L^\infty})G = a(t)G(t)(1 + \ln G(t))\end{aligned}$$

for $a(t) = C(C_0 + \|\omega(t)\|_{L^\infty})$. Then if we let $Y(t) = \ln G(t)$, we get $Y'(t) \le a(t)(1+Y(t))$ (or $(Y+1)'(t) \le a(t)(Y+1)(t)$) which we can use the differential version of Gronwall's lemma on. This yields that

$$Y(t) \le (Y(0) + 1)\exp\left(\int_0^t a(s)\,ds\right) - 1.$$

From this, we can derive the following (double exponential) bound:

$$\|u(t)\|_{H^m} \le \exp\left[(\ln(\|u_0\|_{H^m} + e) + 1)e^{A(t)} - 1\right] - e, \qquad A(t) = \int_0^t a(s)\,ds.$$

Therefore, we would have 2D global regularity if we had an L^∞ bound on the vorticity ω. It turns out that we do indeed have such a bound in two dimensions, which we state in the following lemma. It gives $A(t) \le C(C_0 + \|\omega_0\|_{L^\infty})t$, so the bound on $\|u(t)\|_m$ is uniform up to any finite T_*, and the solution continues globally. □

Lemma 3.0.4 (L^∞ norm bound for vorticity ω)**.** *For classical 2D flows (e.g. ω solving the 2D vorticity equation $\omega_t + v\cdot\nabla\omega = \nu\Delta\omega$, $\nu \ge 0$, bounded velocity and sufficient decay at $|x| \to \infty$), with bounded $\omega_0 = \operatorname{curl} v_0$, we have*

$$|\omega(\cdot,t)|_{L^\infty} \le |\omega_0|_{L^\infty}, \quad \forall\, 0 \le t \le T.$$

Proof. For inviscid flows ($\nu = 0$), this follows from ω being conserved along particle trajectories $X(\alpha,t)$. We have $\omega(X(\alpha,t),t) = \omega_0(\alpha)$, so in fact, $|\omega(\cdot,t)|_{L^\infty} = |\omega_0|_{L^\infty}, \forall 0 \le t \le T$.

For viscous flows ($\nu > 0$), we can rewrite the equation as

$$(\partial_t - \nu\Delta + v\cdot\nabla)\omega = 0.$$

This is a parabolic PDE in non-divergence form, with $c \equiv 0$. Then the inequality $|\omega(\cdot,t)|_{L^\infty} \le |\omega_0|_{L^\infty}, \forall\, 0 \le t \le T$ arises simply from the weak maximum principle for parabolic equations applied to ω and $-\omega$, assuming sufficient decay as $|x| \to \infty$. □

4. Discussion

First, we will discuss what enabled this energy method based proof of global regularity in two dimensions, and why this specific strategy fails in three dimensions. The energy and logarithmic estimates work essentially the same in both cases, but in two dimensions $\|\omega(\cdot,t)\|_{L^\infty}$ stays bounded by its initial value, while in three dimensions we lack this bound. In [3], Tao describes how in three dimensions, a phenomenon called "vortex stretching" comes into play. In an incompressible inviscid flow in 3D we have

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)v,$$

so this term describes stretching and tilting of the vorticity by the velocity gradient. In 2D flows, we just have $\omega \cdot \nabla v = 0$, so we don't have vortex stretching, and we have a maximum principle or bound on the L^∞ norm of ω. This allows us to use the Beale-Kato-Majda Criterion (i.e. Theorem 3.6 in [1] or its 2D variant) to prove global regularity.

Beyond this Beale-Kato-Majda blowup criterion from 1984, there have been several more criterions that say that if some norm of the solution stays bounded, then the solution stays regular (or the contrapositive). Some examples include the following: if a classical 3D Navier-Stokes velocity u from smooth, rapidly decaying divergence-free data first blows up at a finite time T_*, then

- $\limsup_{t\uparrow T_*} \|u(t)\|_{L^3_x(\mathbb{R}^3)} = +\infty$ (Escauriaza-Seregin-Sverak, 2003)
- $\lim_{t\uparrow T_*} \|u(t)\|_{L^3_x(\mathbb{R}^3)} = +\infty$ (Seregin, 2012)
- $\limsup_{t\uparrow T_*} \|u(t)\|_{L^{3,q}_x(\mathbf{R}^3)} = +\infty$ for any $0 < q < \infty$ (Phuc, 2015)

However, none of these so far have been enough to prove global regularity in 3D.

In [2], Tao provides ideas on why 3D regularity has been so difficult, specifically to do with the homogeneity / scale invariance of Navier-Stokes. If we scale via

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

we can show that the L^q norm of u transforms in 3D by a factor of $\lambda^{1-3/q}$:

$$\|u_\lambda(\cdot,t)\|_{L^q} = \lambda^{1-3/q}\|u(\cdot,\lambda^2 t)\|_{L^q}, \qquad 1 \le q \le \infty.$$

Therefore, the $L^3(\mathbb{R}^3)$ norm is scale invariant, while L^q is *supercritical* for $q < 3$ and *subcritical* for $q > 3$. The kinetic energy $\|u\|^2_{L^2}$, for example, then corresponds to a supercritical quantity, so we have very little control of this at finer scales. In fact, [2] says that all known a-priori bounds for 3D Navier-Stokes (e.g. energy) are either supercritical or non-coercive (control of these quantities doesn't really control the solution). Therefore, one strategy for Navier-Stokes global regularity would be to discover a new globally controlled quantity that is both coercive and also critical/subcritical – however, this has not succeeded so far.

One interesting development from 2014 is found in [4] where Tao constructed an "averaged" Navier-Stokes equation in 3D. It obeys the same energy law and essentially the same function-space upper bounds on the nonlinearity, however, Tao was able to construct solutions for it that blow up in finite time. This formalizes the "supercriticality barrier": any proof method for Navier-Stokes that relies only on this energy law and these bounds can't work, since this superficially similar averaged system can blow up. Tao's strategy to construct a solution to the averaged equation that blows up is that of a von Neumann machine; it creates a replica of itself at a finer scale after some delay and mostly erases its original instantiation. Then, time between cascades from each frequency scale to the next decay exponentially, which leads to blow-up at a finite time.

Tao believes that there's a possibility of replicating this behavior in the true Navier-Stokes equations, and one proposed approach is to make the incompressible fluid have the ability to "perform computations" and program a state that self-replicates into a rescaling of itself. So far, it has not been shown that it's possible to implement this kind of computational ability in the Navier-Stokes equations, but it has been shown that for specifically chosen potential functions, the dynamics of particles in a potential well can function as a universal Turing machine, and there have been some results related to this for the Euler equations on a Riemannian manifold (though so far not sufficient to create solutions that blow up.)

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